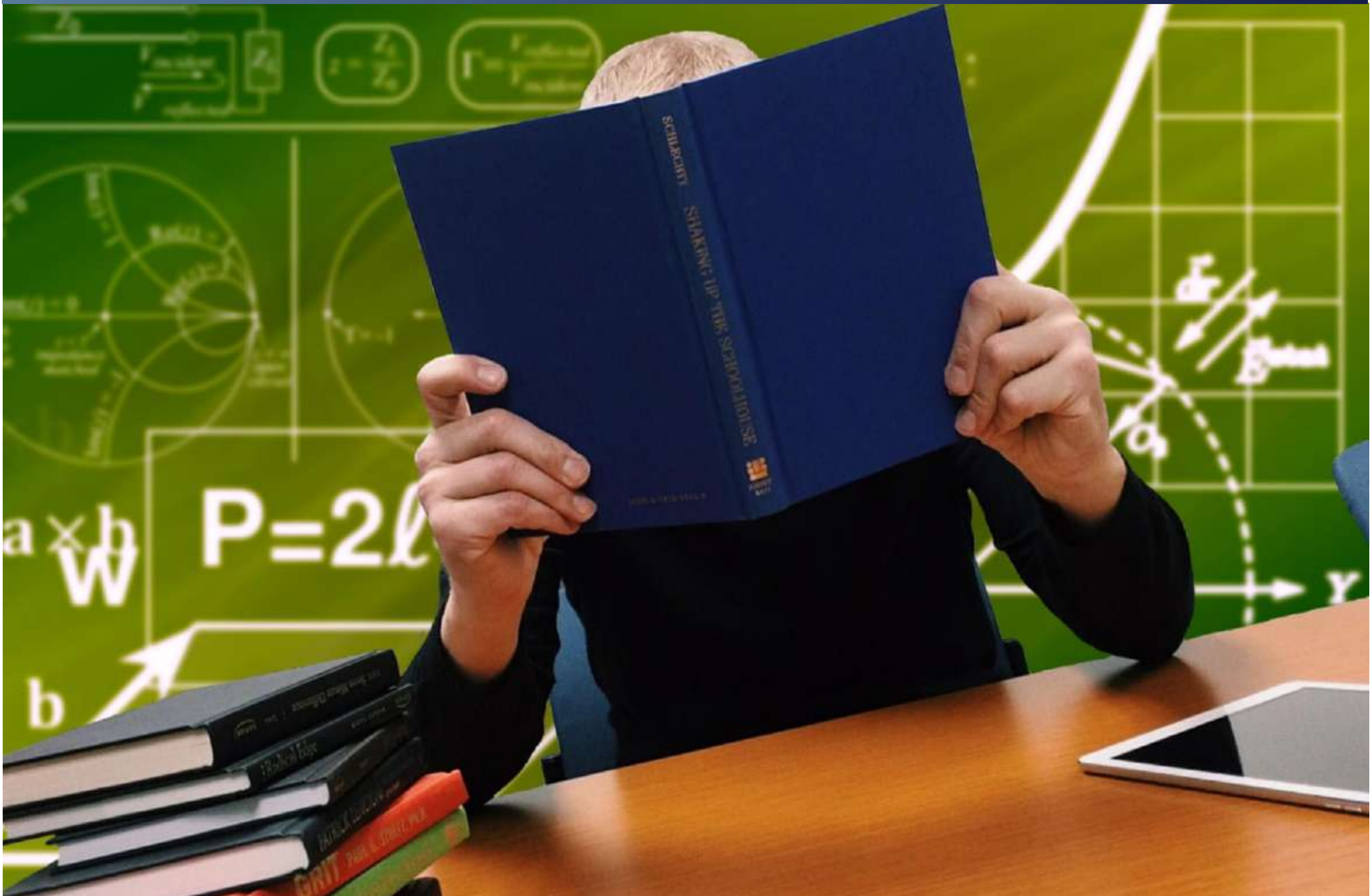
Ejercicios y Talleres



puedes enviarlos a
klasesdematematicasymas@gmail.com

DEPARTAMENTO DE MATEMATICAS

CÁLCULO VECTORIAL

1. Encuentre un punto sobre la curva $\vec{r}(t) = (5\sin t, 5\cos t, 12t)$ que está a una distancia de 24π unidades del origen a lo largo de la curva.
2. Determinar los vectores unitarios $T(t)$ y $N(t)$ para la curva $\vec{r}(t) = (5\sin t, 5\cos t)$ en $t = \frac{\pi}{3}$. Graficar
3. Para la función $f(x,y) = \sqrt{x^2 + y^2 - 6x + 2y + 10}$
   a) Determinar el dominio y hacer un gráfico
   b) Construir curvas de nivel para $c = 0,1,2,-1,-2$
4. Para la función $f(x,y) = e^{-x}Ln(y) - e^{-y}Ln(x)$
   a) Hallar $\frac{\partial z}{\partial x}$ y $\frac{\partial z}{\partial y}$
   b) Demostrar que satisface la ecuación $\frac{\partial^2 Z}{\partial x^2} + \frac{\partial^2 Z}{\partial y^2} = 0$
5. Sean $W(s,t) = F\big(u(s,t), v(s,t)\big)$, donde $u(1,0) = 2$, $u_s(1,0) = -2$, $u_t(1,0) = 6$, $v(1,0) = 3$, $v_s(1,0) = 5$, $v_t(1,0) = 4$, $F_u(2,3) = -1$, $F_v(2,3) = 10$. Encuentre $W_s(1,0)$ y $W_t(1,0)$
6. Para $v = xln(x + r) - r$ donde $r^2 = x^2 + y^2$. Mostrar que $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{1}{x+r}$
   b) Para $z = yf(x^2 - y^2)$ comprobar que $\frac{1}{x}\frac{\partial z}{\partial x} + \frac{1}{y}\frac{\partial z}{\partial y} = \frac{z}{y^2}$ (donde f es una función derivable)

1. $\vec{r}(t) = (5\,sent, 5\cos t, 12t)$

distancia $= 24\pi$

$$d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2} = 24\pi$$

Punto 1 $(0,0,0)$ Origen

Punto 2 $(5\,sent, 5\cos t, 12t)$

$$d = \sqrt{(5\,sent - 0)^2 + (5\cos t - 0)^2 + (12t - 0)^2} = 24\pi$$

$$d = \sqrt{(5\,sent)^2 + (5\cos t)^2 + (12t)^2} = 24\pi$$

$$d = \sqrt{25\,sen^2 t + 25\cos^2 t + 144t^2} = 24\pi$$

$$d = \sqrt{25(sen^2 t + \cos^2 t) + 144t^2} = 24\pi \qquad sen^2 t + \cos^2 t = 1$$

$$d = \sqrt{25 + 144t^2} = 24\pi \qquad 25 + 144t^2 = 24^2\pi^2$$

$$144t^2 = 24^2\pi^2 - 25 \qquad t^2 = \frac{24^2\pi^2 - 25}{144}$$

$$t^2 = 39{,}30 \qquad t = \sqrt{39{,}30} = 6{,}27$$

En $t = 6{,}27$ se cumple la condición. Por tanto el punto sobre la curva es:

$$r(6.27) = (5 \cdot sen\, 6.27,\ 5 \cdot \cos 6.27,\ 12 \cdot 6.27)$$

$$r(6.27) = (-0.069,\ 4.99,\ 75.23)$$

2. $\vec{r}(t) = 5\,sent, 5\cos t$ en $t = \frac{\pi}{3}$

$$T(t) = \frac{r'(t)}{|r'(t)|} \qquad r'(t) = (5\cos t, -5\,sent)$$

$$|r'(t)| = \sqrt{(5\cos t)^2 + (-5\,sent)^2}$$

$$|r'(t)| = \sqrt{25\cos^2 t + 25\,sen^2 t}$$

$$|r'(t)| = \sqrt{25(\cos^2 t + sen^2 t)} = \sqrt{25} = 5$$

$$T(t) = \frac{r'(t)}{|r'(t)|} = \frac{(5\cos t, -5\,sent)}{5} = (\cos t, -sent)$$

$$T(\pi/3) = \cos \pi/3, -sen\, \pi/3 = (0.5, -0.8660)$$

$$r''(t) = (-5\,sent, -5\cos t)$$

$$N(t) = \frac{r''(t) - a_T T}{a_N}$$

$$\cancel{a_T = \frac{r' \cdot r''}{|r'|} = \frac{(5\,sent, 5\cos t)\cdot(-5\,sent, -5\cos t)}{\sqrt{(5\,sent)^2 + (5\cos t)^2}}}$$

$$\cancel{a_T = -25\,sen^2 t}$$

$$a_T = \frac{r' \cdot r''}{|r'|} = \frac{(5\cos t, -5\operatorname{sen} t)\cdot(-5\operatorname{sen} t, -5\cos t)}{\sqrt{(5\cos t)^2 + (-5\operatorname{sen} t)^2}}$$

$$a_T = \frac{-25\cos t \operatorname{sen} t + 25 \operatorname{sen} t \cos t}{\sqrt{25\cos^2 t + 25\operatorname{sen}^2 t}} = \frac{0}{\sqrt{25}} = 0$$

$$a_N = \frac{|r' \times r''|}{|r'|}$$

$$r' \times r'' = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5\cos t & -5\operatorname{sen} t & 0 \\ -5\operatorname{sen} t & -5\cos t & 0 \end{vmatrix} = \hat{k}\left(-25\cos^2 t - 25\operatorname{sen}^2 t\right)$$

$$= -25(\cos^2 t + \operatorname{sen}^2 t)\,\hat{k}$$

$$= -25\hat{k}$$

$$|r' \times r''| = 25$$

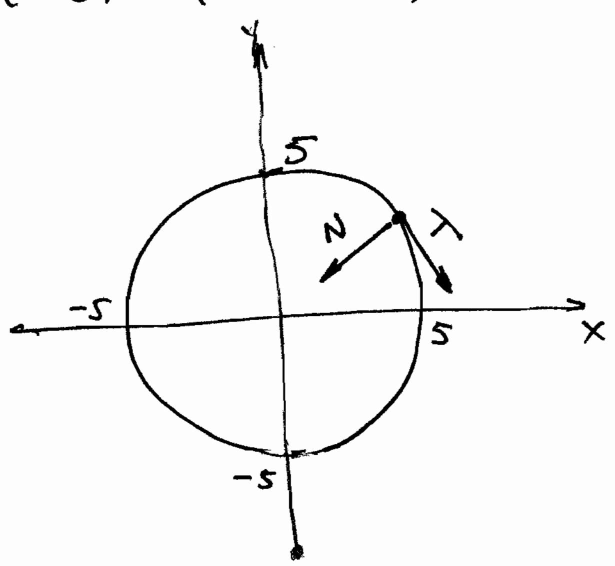
$$|r'| = \sqrt{25} = 5$$

$$a_N = \frac{|r' \times r''|}{|r'|} = \frac{25}{5} = 5$$

$$N = \frac{r''(t) - a_T T}{a_N} = \frac{(-5\operatorname{sen} t, -5\cos t) - 0}{5}$$

$N = (-\operatorname{sen} t, -\cos t)$ $\quad N(\pi/3) = (-\operatorname{sen} \pi/3, -\cos \pi/3)$

$N(\pi/3) = (-0.8660, -0.5)$



$r(\pi/3) = 5\operatorname{sen}\pi/3,\ 5\cos\pi/3$

$= 4.33,\ 2.5$

3. $f(x,y)=\sqrt{x^2+y^2-6x+2y+10}$

Dominio $x^2+y^2-6x+2y+10\geq 0$

$$x^2-6x+y^2+2y+10\geq 0$$

$$x^2-6x+9-\cancel{9}+y^2+2y+1-\cancel{1}+\cancel{10}\geq 0$$

$$(x-3)^2+(y+1)^2\geq 0$$

Dominio todos los puntos (x,y) que cumplen con $(x-3)^2+(y+1)^2\geq 0$ $\Rightarrow$ Todo el plano xy

gráfico $f(x,y)=\sqrt{(x-3)^2+(y+1)^2}$

$$z=\sqrt{(x-3)^2+(y+1)^2}$$

Si $z=1$ $\quad 1=(x-3)^2+(y+1)^2$

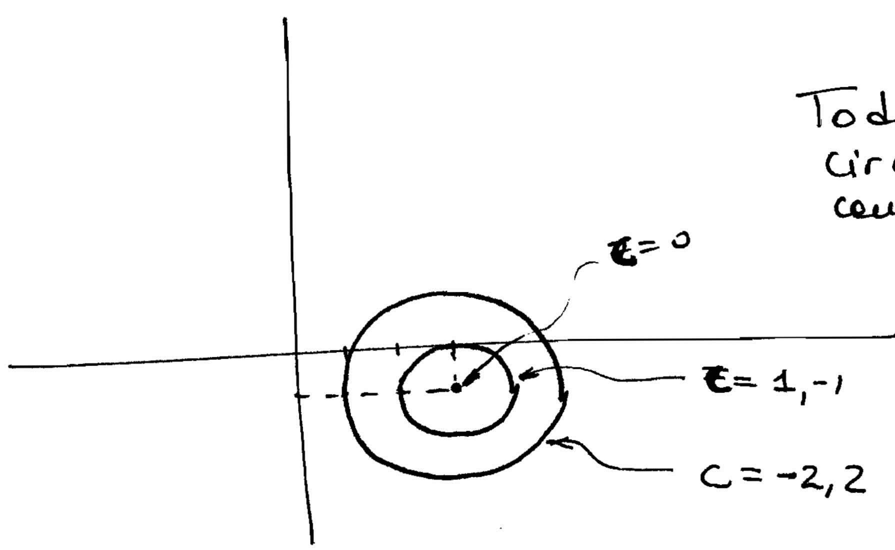
Si $z=0$ $\quad 0=(x-3)^2+(y+1)^2$

Si $z=2$ $\quad 2^2=(x-3)^2+(y+1)^2$

$z=-1$ $\quad (-1)^2=(x-3)^2+(y+1)^2$

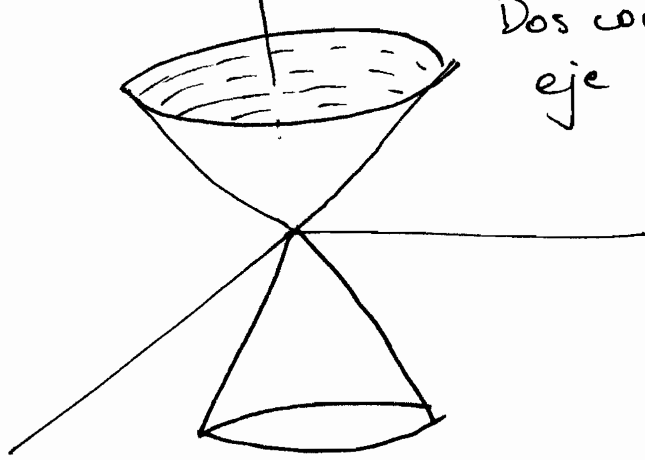
$(-2)^2=(x-3)^2+(y+1)^2$

Para las curvas de nivel se tiene



Todas son circunferencias centradas en $(3,-1)$

El gráfico por tanto es



Dos conos colocados a lo largo del eje z.

4. $f(x,y) = e^{-x}\ln y - e^{-y}\ln x$

a) $\dfrac{\partial z}{\partial x} = -e^{-x}\ln y - e^{-y}\cdot\dfrac{1}{x}$

$\dfrac{\partial z}{\partial y} = e^{-x}\cdot\dfrac{1}{y} + e^{-y}\ln x$

b) $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0$

$\dfrac{\partial^2 z}{\partial x^2} = e^{-x}\ln y + \dfrac{e^{-y}}{x^2}$

$\dfrac{\partial^2 z}{\partial y^2} = -\dfrac{e^{-x}}{y^2} - e^{-y}\ln x$

$\left(e^{-x}\ln y + \dfrac{e^{-y}}{x^2}\right) + \left(-\dfrac{e^{-x}}{y^2} - e^{-y}\ln x\right) = 0$ no se cumple.

5) $W(s,t) = F(u(s,t), v(s,t))$

$u(1,0) = 2 \quad u_s(1,0) = -2 \quad u_t(1,0) = 6 \quad v(1,0) = 3 \quad v_s(1,0) = 5$

$v_t(1,0) = 4$

$F_u(2,3) = -1 \quad F_v(2,3) = 10$

$\omega_s(1,0) \qquad \omega_s = F_u \cdot u_s + F_v \cdot v_s$

$\omega_s(1,0) = F_u(u(1,0), v(1,0)) \cdot u_s(1,0) + F_v(u(1,0), v(1,0)) \cdot v_s(1,0)$

$\omega_s(1,0) = F_u(2,3) \cdot u_s(1,0) + F_v(2,3) \cdot v_s(1,0)$

$= (-1)\cdot(-2) + 10 \cdot 5 = 52$

$\omega_t(1,0) = F_u(u(1,0), v(1,0)) \cdot u_t(1,0) + F_v(u(1,0), v(1,0)) \cdot v_t(1,0)$

$= F_u(2,3) \cdot u_t(1,0) + F_v(2,3) \cdot v_t(1,0)$

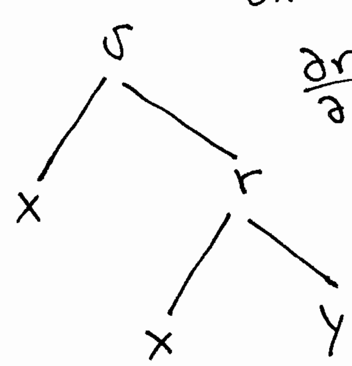
$= (-1)\cdot 6 + 10 \cdot 4 = 34$

6) $v = x\ln(x+r) - r \qquad r^2 = x^2 + y^2$

$\dfrac{\partial r}{\partial x} = \dfrac{2x}{2r} = \dfrac{x}{r}$

$\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$

$\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = \dfrac{1}{x+r}$



$\dfrac{\partial v}{\partial x} = \dfrac{\partial v}{\partial x} + \dfrac{\partial v}{\partial r}\cdot\dfrac{\partial r}{\partial x}$

$\dfrac{\partial v}{\partial x} = \ln(x+r) + \dfrac{x}{x+r} + \left(\dfrac{x}{x+r} - 1\right)\left(\dfrac{x}{r}\right)$

$\dfrac{\partial v}{\partial y} = \dfrac{\partial v}{\partial r}\cdot\dfrac{\partial r}{\partial y} = \left(\dfrac{x}{x+r} - 1\right)\left(\dfrac{y}{r}\right)$

Simplificando

$$\frac{\partial v}{\partial x} = \ln(x+r) + \frac{x}{x+r} + \left(\frac{x-x-r}{x+r}\right)\left(\frac{x}{r}\right)$$

$$= \ln(x+r) + \frac{x}{x+r} - \frac{r}{x+r}\cdot\frac{x}{r} = \ln(x+r) + \frac{x}{x+r} - \frac{x}{x+r} = \underline{\underline{\ln(x+r)}}$$

$$\frac{\partial v}{\partial x} = \ln(x+r)$$

$$\frac{\partial v}{\partial y} = \frac{x-x-r}{x+r}\cdot\frac{y}{r} = \frac{-r}{x+r}\cdot\frac{y}{r} = -\frac{y}{x+r}$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial r}\left(\frac{\partial v}{\partial x}\right)\cdot\frac{\partial r}{\partial x}$$

$$= \frac{1}{x+r} + \frac{1}{x+r}\cdot\frac{x}{r} = \frac{1}{x+r}\left(1+\frac{x}{r}\right) = \frac{1}{x+r}\left(\frac{r+x}{r}\right) = \frac{1}{r}$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial r}\left(\frac{\partial v}{\partial y}\right)\cdot\frac{\partial r}{\partial y}$$

$$= \frac{-1}{x+r} + \frac{y}{(x+r)^2}\cdot\frac{y}{r} = \frac{-(x+r)r + y^2}{(x+r)^2\cdot r} = \frac{-rx-r^2+y^2}{r(x+r)^2}$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{1}{r} + \frac{-rx-r^2+y^2}{r(x+r)^2}$$

$$= \frac{(x+r)^2 - rx - r^2 + y^2}{r(x+r)^2}$$

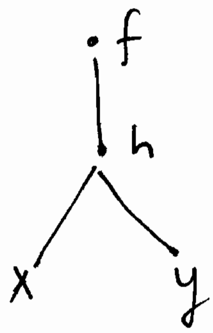
$$= \frac{x^2 + \cancel{2xr} + \cancel{r^2} - \cancel{rx} - \cancel{r^2} + y^2}{r(x+r)^2}$$

$$= \frac{x^2+xr+y^2}{r(x+r)^2} = \frac{x^2+y^2+xr}{r(x+r)^2} = \frac{r^2+xr}{r(x+r)^2}$$

$$= \frac{\cancel{r}(r+x)}{\cancel{r}(x+r)^{\cancel{2}}} = \frac{1}{x+r}$$ { Demostrado.

b) $z = y\,f(x^2-y^2)$

$h = x^2-y^2 \qquad z = y\,f(h)$



$$\frac{\partial z}{\partial x} = y\cdot\frac{\partial f}{\partial h}\cdot\frac{\partial h}{\partial x}$$

$$= y\cdot\frac{\partial f}{\partial h}\cdot 2x = 2xy\,\frac{\partial f}{\partial h}.$$

$$\frac{\partial z}{\partial y} = f(x^2-y^2) + y\,\frac{\partial f}{\partial h}\cdot\frac{\partial h}{\partial y}$$

$$= f(x^2-y^2) + y\,\frac{\partial f}{\partial h}\cdot(-2y)$$

$$\frac{1}{x}\,\frac{\partial z}{\partial x} + \frac{1}{y}\,\frac{\partial z}{\partial y} = \frac{z}{y^2}$$

$$\frac{1}{x}\left(2xy\,\frac{\partial f}{\partial h}\right) + \frac{1}{y}\left(f(x^2-y^2) + y\,\frac{\partial f}{\partial h}(-2y)\right) = \frac{z}{y^2}$$

$$\cancel{2y\,\frac{\partial f}{\partial h}} + \frac{1}{y}\,f(x^2-y^2) + \cancel{\frac{\partial f}{\partial h}(-2y)} = \frac{z}{y^2}$$

$$\frac{1}{y}\,f(x^2-y^2) = \frac{z}{y^2} \qquad f(x^2-y^2) = \frac{z}{y}$$

$$\frac{1}{y}\cdot\frac{z}{y} = \frac{z}{y^2}$$ { Demostrado