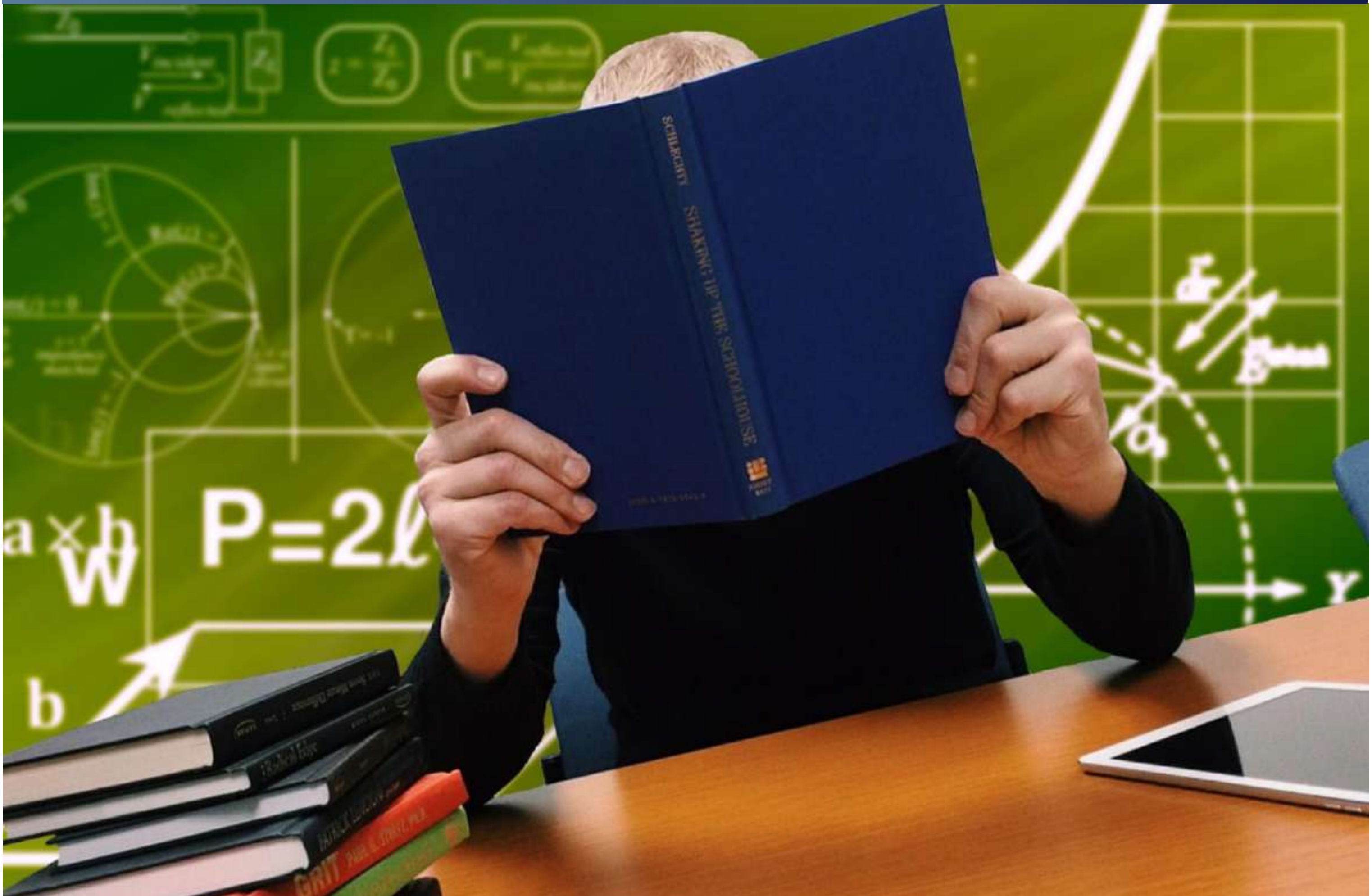
Ejercicios y Talleres



puedes enviarlos a
klasesdematematicasymas@gmail.com

Para cada una de las series siguientes determine si la serie diverge o converge. Justifique su respuesta.

1. $\sum_{n=2}^{\infty}\frac{(-1)^n 2^{n-2}}{3^{n+1}}$

2. $\sum_{n=0}^{\infty}\frac{(-1)^{n+1}3^n}{3^{2n}}$

3. $\sum_{n=0}^{\infty}\frac{7}{n^{3/5}}$

4. $1-\frac{2}{3}+\frac{3}{9}-\frac{4}{27}+\frac{5}{81}-\frac{6}{243}+\cdots$

5. $\sum_{n=0}^{\infty}\frac{\left(\frac{3}{4}\right)^{3n+1}*(n-1)!}{(2n)!}$

En los ejercicios 4 a 6, determine el intervalo de convergencia:

6. $\sum_{n=0}^{\infty}\frac{(-1)^{2n}x^{2n}}{(2n+1)!}$

8. $\sum_{n=0}^{\infty}\frac{(-1)^{n+1}x^{2n+1}}{n}$

Use la serie geométrica para encontrar una serie de potencias que represente a cada una de las funciones en los ejercicios 7 a 9. Determine además el intervalo de convergencia.

9. $f(x)=\frac{1}{1-2x^2}$

10. $f(x)=\frac{1}{5-9x}$

11. $f(x)=\frac{x^2}{2+x^3}$

Use la serie de Maclaurin para hallar los cinco primeros términos de la serie que represente a cada una de las funciones en los ejercicios 7 a 9. Determine además el intervalo de convergencia:

10. $f(x)=sen\left(\frac{x}{4}\right)$

12. $f(x)=x^2e^{\frac{x}{3}}$

13. Evalúe usando series de potencias $ln\,0.8$. Use 6 términos de la serie y compare con el resultado de la calculadora. ¿Cuál es el error relativo?

1. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n 2^{n-2}}{3^{n+1}}$ Por prueba de la razón

$$|a_{n+1}| = \frac{2^{n-2+1}}{3^{n+1+1}} = \frac{2^{n-1}}{3^{n+2}}$$

$$|a_n| = \frac{2^{n-2}}{3^{n+1}}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{\dfrac{2^{n-1}}{3^{n+2}}}{\dfrac{2^{n-2}}{3^{n+1}}} = \lim_{n\to\infty} \frac{2^{n-1}\cdot 3^{n+1}}{2^{n-2}\, 3^{n+2}}$$

$$= 2^{1}\cdot 3^{-1} = \frac{2}{3}$$ la serie converge $\frac{2}{3} < 1$

2. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}\cdot 3^n}{3^{2n}} \Rightarrow$ Por prueba de la razón

$$|a_{n+1}| = \frac{3^{n+1}}{3^{2(n+1)}} = \frac{3^{n+1}}{3^{2n+2}}$$

$$|a_n| = \frac{3^n}{3^{2n}} \qquad \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{\dfrac{3^{n+1}}{3^{2n+2}}}{\dfrac{3^n}{3^{2n}}} = \frac{3^{n+1}\cdot 3^{2n}}{3^{2n+2}\cdot 3^n}$$

$$= \frac{3}{3^2} = \frac{1}{3} < 1$$ Converge

3) $\displaystyle\sum_{n=0}^{\infty} \frac{7}{n^{3/5}} = \sum_{n=0}^{\infty} 7\, n^{-3/5}$ ~~Es una serie geométrica por tanto converge~~

~~Su suma es~~ $\cancel{S = \frac{a}{1-r}}$ $\cancel{S = \frac{7}{1-(-3/5)} = \frac{7}{1+\frac{3}{5}} = \frac{7}{\frac{8}{5}} = \frac{35}{8}}$

Tambien se puede mirar por

$$\lim_{n\to\infty} \frac{7}{n^{3/5}} = 0$$ converge.

(4) $1 - \frac{2}{3} + \frac{3}{9} - \frac{4}{27} + \frac{5}{81} - \frac{6}{243} + \cdots$

Se puede escribir la serie como

$$\sum_{n=0}^{\infty} \frac{n+1}{3^n} \cdot (-1)^n = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{n+1}{3^n}$$

Utilizamos criterio de la razón

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{\frac{n+1+1}{3^{n+1}}}{\frac{n+1}{3^n}} \Rightarrow \lim_{n\to\infty} \frac{(n+2)\,3^n}{(n+1)\cdot 3^{n+1}} = \lim_{n\to\infty} \frac{(n+2)}{(n+1)\cdot 3}$$

$$= \frac{1}{3} \lim_{n\to\infty} \frac{n+2}{n+1} = \frac{1}{3} \lim_{n\to\infty} \frac{\frac{n}{n}+\frac{2}{n}}{\frac{n}{n}+\frac{1}{n}} = \frac{1}{3} \lim_{n\to\infty} \frac{1+\frac{2}{n}}{1+\frac{1}{n}} = \frac{1}{3}\cdot 1 = \frac{1}{3}$$

la serie converge

(5) $$\sum_{n=0}^{\infty} \frac{\left(\frac{3}{4}\right)^{3n+1} \cdot (n-1)!}{(2n)!}$$

Por criterio de Razón

$$a_{n+1} = \frac{\left(\frac{3}{4}\right)^{3(n+1)+1} \cdot (n+1-1)!}{(2(n+1))!} = \frac{\left(\frac{3}{4}\right)^{3n+4} \cdot n!}{(2n+2)!}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{\frac{\left(\frac{3}{4}\right)^{3n+4} \cdot n!}{(2n+2)!}}{\frac{\left(\frac{3}{4}\right)^{3n+1} \cdot (n-1)!}{(2n)!}} = \lim_{n\to\infty} \frac{\left(\frac{3}{4}\right)^{3n+4} \cdot n! \cdot 2n!}{(2n+2)! \cdot \left(\frac{3}{4}\right)^{3n+1} \cdot (n-1)!}$$

$$= \lim_{n\to\infty} \frac{\cancel{\left(\frac{3}{4}\right)^{3n+1}} \cdot \left(\frac{3}{4}\right)^3 \cdot n \cdot \cancel{(n-1)!} \cdot \cancel{(2n)!}}{(2n+2)(2n+1)\cancel{(2n)!} \cdot \cancel{\left(\frac{3}{4}\right)^{3n+1}} \cdot \cancel{(n-1)!}}$$

$$= \lim_{n\to\infty} \frac{\left(\frac{3}{4}\right)^3 n}{(2n+2)(2n+1)} = \lim_{n\to\infty} \frac{\frac{27}{64} n}{4n^2+6n+2}$$

$$= \lim_{n\to\infty} \frac{27}{64} \frac{n}{4n^2+6n+2} = \lim_{n\to\infty} \frac{27}{64} \cdot \frac{\frac{n}{n^2}}{\frac{4n^2}{n^2}+\frac{6n}{n^2}+\frac{2}{n^2}}$$

$$\frac{27}{64} \lim_{n\to\infty} \frac{\frac{1}{n}}{4+\frac{6}{n}+\frac{2}{n^2}} = \frac{27}{64} \cdot 0 = 0$$ la serie converge.

6) $$\sum_{n=0}^{\infty} \frac{(-1)^{2n} x^{2n}}{(2n+1)!}$$

Usamos criterio de la razón

$$a_{n+1} = \frac{x^{2n+2}}{(2n+2+1)!} = \frac{x^{2n+2}}{(2n+3)!}$$

$$\lim_{n\to\infty} \frac{\frac{x^{2n+2}}{(2n+3)!}}{\frac{x^{2n}}{(2n+1)!}} = \lim_{n\to\infty} \frac{x^{2n+2}\cdot(2n+1)!}{(2n+3)!\,x^{2n}}$$

$$= \lim_{n\to\infty} \frac{\cancel{x^{2n}}\cdot x^2\cdot \cancel{(2n+1)!}}{(2n+3)(2n+2)\cancel{(2n+1)!}\,\cancel{x^{2n}}} = \lim_{n\to\infty} \frac{x^2}{(2n+3)(2n+2)}$$

$$= \lim_{n\to\infty} \frac{1}{4n^2+10n+6}\cdot x^2 = x^2 \lim_{n\to\infty} \frac{1}{0} = x^2\cdot 0 = 0$$

Por tanto converge para todo x.

7) $$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}\cdot x^{2n+1}}{n}$$

$$a_{n+1} = \frac{x^{2n+3}}{n+1} \qquad \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{\frac{x^{2n+3}}{n+1}}{\frac{x^{2n+1}}{n}}$$

$$= \lim_{n\to\infty} \frac{x^{2n+3}\cdot n}{(n+1)(x^{2n+1})} = \lim_{n\to\infty} \frac{x^2\cdot n}{n+1}$$

$$= x^2 \lim_{n\to\infty} \frac{n}{n+1} = x^2 \lim_{n\to\infty} \frac{\frac{n}{n}}{\frac{n}{n}+\frac{1}{n}} = x^2 \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}}$$

$$= x^2\cdot 1 = x^2$$

Ahora $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$ Para converger Por tanto.

$$x^2 < 1 \qquad -1 < x < 1$$

Radio de convergencia $(-1, 1)$

8) $f(x) = \dfrac{1}{1-2x^2}$

Series geométrica =

$\sum_{k=0}^{\infty} a \cdot r^k = \dfrac{a}{1-r}$ Para nuestro caso $a = 1$, $r = (2x^2)$

$$\frac{1}{1-2x^2} = \sum_{k=0}^{\infty} 1(2x^2)^k = 1 + (2x^2)^1 + (2x^2)^2 + (2x^2)^3 + \cdots$$

$$= 1 + 2x^2 + 4x^2 + 8x^6 + 16x^8 + \cdots$$

Radio de convergencia

$2x^2 < 1 \qquad x^2 < \frac{1}{2} \qquad \frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$

$-\frac{\sqrt{2}}{2} < x < \frac{\sqrt{2}}{2}$

9) $f(x) = \dfrac{1}{5-9x} = \dfrac{1}{5\left(1-\frac{9}{5}x\right)} = \dfrac{\frac{1}{5}}{1-\frac{9}{5}x}$

$a = \frac{1}{5} \qquad r = \frac{9}{5}x$

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{5}\cdot\left(\frac{9}{5}x\right)^n = \frac{1}{5} + \frac{1}{5}\left(\frac{9}{5}x\right) + \frac{1}{5}\left(\frac{9}{5}x\right)^2 + \frac{1}{5}\left(\frac{9}{5}x\right)^3 + \frac{1}{5}\left(\frac{9}{5}x\right)^4 +$$

$$= \frac{1}{5} + \frac{9}{25}x + \frac{81}{125}x^2 + \frac{729}{625}x^3 + \frac{6561}{3125}x^4 + \cdots$$

Radio de convergencia

$\left|\frac{9}{5}x\right| < 1 \qquad |x| < \frac{5}{9} \qquad -\frac{5}{9} < x < \frac{5}{9}$

10) $f(x) = \dfrac{x^2}{2+x^3} = x^2 \cdot \dfrac{1}{2+x^3} = x^2 \cdot \dfrac{1}{2\left(1+\frac{1}{2}x^3\right)} = x^2 \cdot \dfrac{\frac{1}{2}}{1+\frac{1}{2}x^3}$

$a = \frac{1}{2} \qquad r = \frac{1}{2}x^3$

$$f(x) = x^2 \cdot \sum_{n=0}^{\infty} \frac{1}{2}\cdot\left(\frac{1}{2}x^3\right)^n = x^2\left[\frac{1}{2} + \frac{1}{2}\left(\frac{1}{2}x^3\right)^1 + \frac{1}{2}\left(\frac{1}{2}x^3\right)^2 + \cdots\right.$$

$$= x^2\left[\frac{1}{2} + \frac{1}{4}x^3 + \frac{1}{8}x^6 + \frac{1}{16}x^9 + \frac{1}{32}x^{12} + \cdots\right]$$

$$= \frac{1}{2}x^2 + \frac{1}{4}x^5 + \frac{1}{8}x^8 + \frac{1}{16}x^{11} + \frac{1}{32}x^{14} + \cdots$$

Radio de convergencia

$\left|\frac{1}{2}x^3\right| < 1 \qquad |x^3| < 2$

$-2 < x^3 < 2$

$\sqrt[3]{-2} < x < \sqrt[3]{2}.$

11. $f(x) = \operatorname{sen}\left(\frac{x}{4}\right)$

$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{IV}(0)}{4!}x^4 + \cdots$

$f(x) = \cancel{\operatorname{sen} 0}^{\,0} + \frac{1}{4}\cos 0 . x - \cancel{\frac{1}{16}\operatorname{sen} 0}^{\,0} . \frac{x^2}{2!} + -\frac{1}{64}\cos 0 \ \frac{x^3}{3!} + \frac{1}{256}\operatorname{sen} 0 . \frac{x^4}{4!} + \frac{1}{1024}\cos 0 . \frac{x^5}{5!}$

$f(x) = \operatorname{sen}\frac{x}{4}$

$f'(x) = \frac{1}{4}\cos\frac{x}{4} \qquad f''(x) = \frac{-1}{16}\operatorname{sen}\frac{x}{4} \qquad f'''(x) = -\frac{1}{64}\cos\frac{x}{4}$

$f^{IV}(x) = \frac{1}{256}\operatorname{sen}\frac{x}{4} \qquad f^{V}(x) = \frac{1}{1024}\cos\frac{x}{4}$

$f(x) = 0 + \frac{1}{4}x - \frac{1}{64}\frac{x^3}{3!} + \frac{1}{1024}\frac{x^5}{5!}$ Para toda x

12) $f(x) = x^2 e^{x/3}$

$g(x) = e^{x/3} \qquad g'(x) = \frac{1}{3}e^{x/3} \qquad g''(x) = \frac{1}{9}e^{x/3}$

$g'''(x) = \frac{1}{27}e^{x/3} \qquad g^{IV}(x) = \frac{1}{81}e^{x/3} \qquad g^{V}(x) = \frac{1}{243}e^{x/3}$

Desarrollo de la función en series de potencias.

$f(x) = f(0) + f'(0)\cdot x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + f^{IV}(0)\frac{x^4}{4!} + f^{V}(0)\frac{x^5}{5!}$

$g(x) = e^0 + \frac{1}{3}e^0\cdot x + \frac{1}{9}e^0\cdot\frac{x^2}{2!} + \frac{1}{27}e^0\cdot\frac{x^3}{3!} + \frac{1}{81}e^0\cdot\frac{x^4}{4!} + \frac{1}{243}e^0\cdot\frac{x^5}{5!}$

$g(x) = 1 + \frac{1}{3}x + \frac{1}{18}x^2 + \frac{1}{162}x^3 + \frac{1}{1944}x^4 + \frac{1}{29160}x^5 + \cdots$

Como la función es $x^2 e^{x/3}$ queda

$$f(x) = x^2 \cdot g(x) = x^2 + \frac{1}{3}x^3 + \frac{1}{18}x^4 + \frac{1}{162}x^5 + \frac{1}{1944}x^6 + \frac{1}{29160}x^7 + \cdots$$

Intervalo de convergencia Todos los reales.

13. $\ln 0.8$

El desarrollo en series de potencias es:

$$f(x) = \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6}$$

Para este caso $x = -0.2$; $1 - 0.2 = 0.8$.

$$\ln(1-0.2) = (-0.2) - \frac{(-0.2)^2}{2} + \frac{(-0.2)^3}{3} - \frac{(0.2)^4}{4} + \frac{(-0.2)^5}{5} - \frac{(-0.2)^6}{6}$$

$$= -0{,}22314{,}1333$$

Usando calculadora $\ln 0.8 = -0.22314355{,}1$

Error Relativo $\dfrac{|-0.223143551 + 0.2231413331|}{0.223143551} \times 100\%$

$= 0{,}00099\%$ de Error relativo.